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Date: December 10, 2024, 2pm–5pm, Evans 891

1 Major Topic: Algebraic Number Theory (Algebra)

References: Neukirch, *Algebraic Number Theory*, Chapters I and II, except II.6. Milne, *Class Field Theory*, Chapters I.1, V.

- Algebraic Integers: integrality, trace and norm, discriminant, Dedekind domains and unique factorization of ideals, Minkowski theory, class group/number, Dirichlet's unit theorem, ramification theory, cyclotomic fields.
- Valuations: p-adic numbers, valuations and absolute values, completions, Hensel's lemma, local and global fields, extensions of valuations, ramification of extensions, Galois theory of valuations.
- Class Field theory: adèles and idèles, statements of local and global class field theory (Artin reciprocity and existence theorems), statement of Chebotarev density.

2 Major Topic: Algebraic Geometry (Algebra)

References: Vakil, *The Rising Sea* (Feb 21, 2024 edition), Chapters 1-19, 21, 23, 25.1, excluding starred sections.

- Schemes: presheaves and sheaves, Spec and Proj, irreducible, reduced, integral, (locally) Noetherian, regularity, smoothness, dimension.
- Morphisms: open/closed/immersions, (locally) finite type, finite, separated, proper, projective, quasicompact, quasiseparated, normalization.
- Representable functors: functor of points, fiber products, morphisms to projective space.
- Sheaves of modules: (quasi)coherent sheaves, line and vector bundles, (very) ampleness, morphisms to projective space, relative Spec and Proj.
- Divisors: Weil divisors, Picard group.
- Differentials: relative differentials, smoothness.

- Cohomology: Čech cohomology of quasicoherent sheaves, statement of Serre duality, derived functors, higher direct images, statement of cohomology and base change.
- Curves: genus, Riemann-Roch, Riemann-Hurwitz, elliptic and hyperelliptic curves, canonical embedding.

3 Minor Topic: Complex Analysis (Analysis)

References: Stein & Shakarchi, *Complex Analysis*, Chapters 1-3, 5-8, except 8.4.

- Cauchy's theorem and integration: Goursat's theorem, Cauchy's theorem/formula, power series, Liouville's theorem, Morera's theorem, analytic continuation.
- Meromorphic functions: Laurent series, residue theorem, argument principle, Rouché's theorem, open mapping theorem, maximum modulus principle, branch cuts.
- Conformal mappings: Möbius transformations, Schwarz lemma, automorphisms of $\mathbf{D}$ and $\mathbf{H}$, Riemann mapping theorem.
- Special functions: infinite products, functions with prescribed zeroes and poles, gamma and zeta functions.

Here is a rough transcript of the exam:

Algebraic Number Theory:

[S] Define a Dedekind domain.

[S] Define the class group of a Dedekind domain. What is a fractional ideal? (fractional ideals mod principal ideals, nonzero R-submodules M of F such that there is nonzero $r \in R$ with $rM \subseteq R$)

[S] Define ring of integers of a number field, what can you say about the class group of the ring of integers? (finite, sketched proof via every class has an integral ideal of norm at most the Minkowski bound)

[T] Compute class group of $\mathbf{Q}(\sqrt{5})$ (this one was strange since the Minkowski bound immediately shows $\mathrm{Cl}(K) = 1$)

[T] State Chebotarev density. Define unramified, Frobenius morphism.

[V] What is the density of a set of primes? (I forgot the defintion of Dirichlet density, only remembering it after the exam, so I stated the natural density $\lim_{n\to\infty} |\mathfrak{p} \in S : N\mathfrak{p} \leq n| / |\mathfrak{p} : N\mathfrak{p} \leq n|$)

[T] How do natural density and Dirichlet density relate? (I forgot so I offered that Chebotarev density is easier to prove for Dirichlet density. Yunqing told me that natural density is stronger than Dirichlet density—if it exists then so does Dirichlet density and they are equal)

[T] Apply Chebotarev density to $\mathbf{Q}(\sqrt{5})/\mathbf{Q}$. Which primes split and what proportion? (use Dedekind-Kummer, for primes not 2 or 5 look at splitting of $x^2 - 5$, then use quadratic reciprocity)

[V] Follow-up to previous: sketch a proof of quadratic reciprocity. (embed quadratic field into $\mathbf{Q}(\zeta_p)$, look at restriction of Frobenius since there's only one map from $(\mathbf{Z}/p\mathbf{Z})^\times$ to $\mathbf{Z}/2\mathbf{Z}$)

Algebraic Geometry:

[S] Is there an analogue of the class group for a Dedekind domain in geometry? Define Picard group of a scheme. ($\mathrm{Cl}(R) \cong \mathrm{Pic}\,\mathrm{Spec}(R)$).

[S] Set up your definition of variety. (integral, separated, finite type over a field. More assumptions to be added later if needed, like regular)

[S] Follow-up to previous: which of these properties continues to hold after extending the base field? (integrality fails—gave standard example of Spec($\mathbf{C}$) as an $\mathbf{R}$-variety)

[T] Compute the Picard group of any variety you want. (I did $\mathbf{P}^1$ using explicit transition functions over the two patches of $\mathbf{P}^1$. Defined $O(n)$ along the way)

[T] Compute Pic($\mathbf{P}^1 \times \mathbf{P}^1$). (I flubbed this one even though I had done it in practice. Followed the method of II.6.6.1 in Hartshorne using the excision sequence $\mathbf{Z} = \mathrm{Cl}(\mathbf{P}^1) \to \mathrm{Cl}(\mathbf{P}^1 \times \mathbf{P}^1) \to \mathrm{Cl}(\mathbf{A}^1 \times \mathbf{P}^1) \to 0$ by excising $Z = \{\infty\} \times \mathbf{P}^1$, so the first map is pullback along the projection to the first $\mathbf{P}^1$. Forgot why the first map is injective until I realized restricting $n[Z]$ to the first copy of $\mathbf{P}^1$ is $n[\infty]$, so can't be trivial.)

[S] During the above: define Weil divisors, divisor class group, correspondence between Weil divisors and invertible sheaves on (factorial) schemes, say why $\mathrm{Pic}(\mathbf{A}^1 \times \mathbf{A}^1) = 0$. (because it is a UFD)

[V] Define smoothness—luckily I was allowed to just define it for varieties over a field. Use Jacobian criterion. (smooth at a point if Jacobian of local equations at that point has corank—calculated in $\kappa(p)$—equal to dimension of the variety)

[V] State Riemann-Roch for curves and give an application. (stated it and Serre duality, computed $\deg \omega_C = 2g - 2$)

[V] Why is $h^0(C, \mathcal{O}_C) = 1$? (Base change to $\overline{k}$ and proved that global sections on proper reduced connected $\overline{k}$-scheme are constants) What fact are you using and why does it hold? (that $H^i(C, L) \otimes_k K \cong H^i(C_K, L_K)$, which holds more generally, for any field extension K/k. I forgot how to prove this in a straightforward way, but they were happy with me saying cohomology commutes with flat base change)

(Probably there was another question somewhere during the AG section, but I have already forgotten it.)

Complex Analysis:

[Z] What can you say about a doubly periodic entire function? (constant by Liouville)

[Z] What if it has a single pole in a fundamental domain? (residue is 0 by integrating around fundamental parallelogram)

[Z] Say I have a function analytic in a punctured disk at 0, what can you say about behavior at 0? Define each term. (can be removable if bounded, pole if $1/f$ approaches 0 at 0, or essential if $f(z) = e^{1/z}$)

[Z] What can you say about behavior near 0 if the singularity is essential? (proved Casorati-Weierstrass)

[Z] State Riemann mapping theorem. Why can't the domain be all of $\mathbf{C}$? (Liouville)

[Z] Give a biholomorphism between an infinite vertical strip and $\mathbf{D}$. (rotate to a horizontal strip, scale and apply exp to get the upper half plane, and use $(i-z)/(i+z)$).

[T] State and prove maximum modulus principle. (essentially the open mapping theorem: use Rouché Theorem—if $f(z) = w$ then can solve $f(z) = w'$ for any w' sufficiently near w' by writing $f(z) - w' = (f(z) - w) + (w - w')$)

[T] If f has no zeroes on a domain, what can you say about $\log|f|$? (wrote Jensen's formula and said that if there are no zeroes then $\log|f|$ satisfies the mean-value property and is harmonic)

Final thoughts: the committee was very nice, and the exam was shorter than I expected (only a bit over 2 hours, had prepared for 3). I forgot a few definitions and basic questions but the professors were very accommodating to let me think for a bit at the board. I definitely felt a bit overprepared for the exam, since there were lots of things that I had prepared that weren't asked about (e.g. classification of low degree/genus curves, local fields, applications of class field theory, etc.). I was also quite nervous about the exam but weekly preparation meetings with my advisor Yunqing helped, and overall the process of preparation (and worrying about being held accountable) was very useful training. Looking back (after having passed), forcing myself to relearn basic things and work through many explicit examples carefully definitely helped my research skills a lot.